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Exact Tracer Diffusion Coefficient in the Asymmetric Random Average Process

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We study tracer diffusion in the continuous-time asymmetric random average process which is an interacting particle system on $\mathbb{R}$ generalizing the Hammersley process. From the equations of motion for the particle-position correlations we obtain the exact tracer diffusion coefficient which is in agreement with a recent heuristic result by Krug and Garcia.

KEY WORDS: Interacting particle systems; random average process; tracer diffusion.

Recent work on interacting particle systems far from equilibrium has focussed on lattice models such as the asymmetric exclusion process and other lattice gas systems.[1–3] Comparatively little is known about particle systems defined on the real line which have appeared e.g., in the context of traffic flow,[4] force propagation in granular media[5] and interface fluctuations.[6] Closely related to the models of refs. 5, 6 is the continuous-time version of the asymmetric random average process studied recently by Krug and Garcia.[7] In this model, a generalization of the Hammersley process,[8] point particles on $\mathbb{R}$ jump with constant rate 1 from position x_i to the right to $x_i + \delta_i$ where δ_i is a random fraction of the headway

$$u_i = x_{i+1} - x_i \tag{1}$$

The moves occur in continuous time, i.e., each particle carries its intrinsic exponential clock: When the clock rings (after an exponentially distributed

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random time with parameter 1), the move is executed. The random jump length δ_i is chosen according to a probability density

$$f_i(\delta_i) = u_i^{-1} \phi(\delta_i/u_i) \tag{2}$$

normalized to $\int_0^1 dr\, \phi(r) = 1$.

In ref. 7 it was shown that the stationary two-point headway correlation function $\langle u_i u_j \rangle$ of this model factorizes for $i \neq j$. Moreover, for $i = j$ the second moment $\langle u^2 \rangle$ of the headway distribution is given by

$$\langle u^2 \rangle = \frac{\mu_1}{\rho^2(\mu_1 - \mu_2)} \tag{3}$$

where $\rho = 1/\langle u \rangle$ is the stationary particle density and

$$\mu_n = \int_0^1 dr\, r^n \phi(r) = \frac{1}{u^{n+1}} \int_0^u dr\, r^n \phi(r/u) \tag{4}$$

are the moments of the jump length distribution.

In order to determine the statistical properties of a tracer particle we introduce the time-dependent joint probability densities $P_{i_1,\ldots,i_k}(x_{i_1},\ldots,x_{i_k})$ of finding the particles with label i_j on positions x_{i_j}. For notational simplicity the dependence on time (and on the initial distribution) is dropped. The mean position $\langle X_i \rangle$ of a tracer particle i is then given by

$$\langle X_i \rangle = \int_{-\infty}^{\infty} dx\, x P_i(x) \tag{5}$$

This yields the stationary drift velocity

$$v = \lim_{t \to \infty} \frac{d}{dt} \langle X_i \rangle \tag{6}$$

In a similar fashion the tracer diffusion coefficient is obtained from the asymptotic mean square displacement

$$D = \lim_{t \to \infty} \frac{d}{dt} (\langle X_i^2 \rangle - \langle X_i \rangle^2) \tag{7}$$

These quantities do not depend on i. For the velocity one finds $v = \mu_1/\rho$.[7] The main result of this paper is the exact derivation of the steady-state diffusion coefficient

$$D = \frac{\mu_1 \mu_2}{\rho^2(\mu_1 - \mu_2)} \tag{8}$$

obtained also by Krug and Garcia using two independent heuristic arguments which lead to an effective Langevin equation for the motion of the tracer particle and an independent-jump approximation respectively.

The key ingredient in calculating v and D is the master equation obeyed by the joint probability densities $P_{i_1,\dots,i_k}(x_{i_1},\dots,x_{i_k})$. E.g., for $k=1$ one has

$$\frac{d}{dt}P_i(x) = -P_i(x) + \int_0^\infty dy_1 \int_0^\infty dy_2 \frac{1}{y_1+y_2} \times \phi\left(\frac{y_1}{y_1+y_2}\right) P_{i,i+1}(x-y_1, x+y_2) \tag{9}$$

The negative contribution results from the particle hopping away from x, while the positive part counts all possibilities of jumping from a position $x-y_1$ to x in the interval $[x-y_1, x+y_2)$ between particles i and $i+1$. Analogously one finds expressions for higher order joint probability densities.

From the joint probability densities one can calculate the expectation values $\langle X_{i_1}\cdots X_{i_k}\rangle$. The key observation necessary for calculating D is the fact that the equations of motion for these expectation values form a closed set for each level k. E.g., for $k=1$ one finds $d/dt\langle X_i\rangle = \mu_1(\langle X_{i+1}\rangle - \langle X_i\rangle)$ which immediately yields the stationary tracer velocity $v=\mu_1/\rho$. The extension to higher order correlation functions is rather tedious, but straightforward. Of particular interest is the quantity

$$C_{i,j}(t) = \langle X_i X_j\rangle - \langle X_i\rangle\langle X_j\rangle \tag{10}$$

After a lengthy sequence of manipulations of integrals involving shifting integration intervals and interchanging the order of integration we find

$$\frac{d}{dt}C_{i,j} = \mu_1[C_{i,j+1} + C_{i+1,j} - 2C_{i,j}] + \mu_2\langle u_i^2\rangle\,\delta_{i,j} \tag{11}$$

with the Kronecker symbol $\delta_{i,j}=1$ for $i=j$ and 0 else. This yields the time derivative of the mean square displacement $d/dt\, C_{i,i} = \mu_2\langle u_i^2\rangle + 2\mu_1(C_{i,i+1} - C_{i,i})$ and hence an expression for the diffusion coefficient.

To calculate D we use $C_{i,i+1} - C_{i,i} = \langle X_i u_i\rangle - \langle X_i\rangle\langle u_i\rangle$ and therefore

$$\begin{aligned} C_{i,i+1} - C_{i,i} &= C_{i-1,i+1} - C_{i-1,i} + \langle u_{i-1}u_i\rangle - \langle u_{i-1}\rangle\langle u_i\rangle \\ &= C_{i-r,i+1} - C_{i-r,i} + \sum_{k=1}^{r} \langle u_{i-k}u_i\rangle - \langle u_{i-k}\rangle\langle u_i\rangle \end{aligned} \tag{12}$$

In the steady state the headway correlations vanish. We conclude that for all particle pairs $(i-r, i)$ the difference $C^*_{i-r,i+1} - C^*_{i-r,i}$ of stationary correlation functions is equal and vanishes: $C^*_{i,i+1} - C^*_{i,i} \equiv \lim_{t\to\infty} \langle X_i u_i \rangle - \langle X_i \rangle \langle u_i \rangle = \lim_{t\to\infty} \langle X_{i-r} u_i \rangle - \langle X_{i-r} \rangle \langle u_i \rangle = 0$. Equation (11) then yields $D = \mu_2 \langle u_i^2 \rangle$ and with (3) the main result (8).

The same result could be obtained in a technically more involved manner by explicitly solving (11) for the type of initial distribution envisaged here, i.e., where $\langle u_i \rangle$ and $\langle u_i u_j \rangle$ take their stationary values. This directly yields the steady-state diffusion coefficient $D = \lim_{t\to\infty} (\langle X_i^2 \rangle - \langle X_i \rangle^2)/t$. Notice that the assumption of stationarity of the one-point and two-point headway correlation function does not imply that the measure itself is stationary.[9]

Since the exact diffusion coefficient (8) agrees with the expression obtained from the independent jump approximation[7] one may wonder whether this approximation is not actually exact as in the case of the totally asymmetric simple exclusion process (TASEP).[10] In the independent-jump approximation the stationary motion of the tracer particle is regarded as a Poisson process. A possible strategy to address this question is the following. We first note that the motion of the tracer particle i is, at all times, independent of the motion of all particles $i-r$ to its left. Hence one may study the semi-infinite system with particle i at its left boundary. Without loss of generality we take $i=1$. Next we define the process in terms of the particle headways u_i where $i \geqslant 1$. In the context of the TASEP this leads to a totally asymmetric zero-range process where particles move to the left and absorption of particles takes place at the left boundary site 1. Each absorption event corresponds to a single move of the tracer particle. Here we are led to a stick representation[11] of the ARAP where u_i represents the length of a stick located on the integer lattice. In each move a fraction δ_i of stick i is broken off and added to stick $i-1$. The motion of the tracer particle corresponds to the absorption at the left boundary of a piece δ_1 of the first stick which takes place after an exponentially distributed random time. Since in the ARAP a jump attempt always succeeds the random time has mean 1. In the steady state the loss δ_1 (i.e., the hopping distance of the tracer particle) is a random variable distributed according to the density $f^*(\delta_1) = \int_0^\infty du\, u^{-1} \phi(\delta_1/u)\, P^*(u)$ where $P^*(u)$ is the stationary headway distribution of the ARAP. If all consecutive hopping increments $\delta_1^{(i)}$ would be independent random variables the steady state motion of the tracer particle would a Poisson process with (random) hopping distance δ_1. From this one recovers the drift velocity v (6) and the diffusion coefficient (8). Independence remains an open question. The factorization of the headway correlations may possibly give a clue as to why the diffusion coefficient comes out correctly from the independent-jump approximation.

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